Momentum conservation and collisions

Impulse

1 Incident momentum is 12 kg m s^{-1}. Rebound momentum is –8 kg m s^{-1}. Impulse on ball is –20 kg m s^{-1}. Impulse on post is +20 kg m s^{-1}.

2 (a) Incident momentum is 1.40×10^{-23} kg m s^{-1}. Rebound momentum is -1.40×10^{-23} kg m s^{-1}. Impulse on wall is $+2.79 \times 10^{-23}$ kg m s^{-1}. Force on wall is

(b) Force on wall is 27.9 kN.

Conservation of momentum and energy.

3 Consider a small time interval *dt* at the start of which the rocket has mass *m* and velocity *v* and at the end of which it has mass *m* + *dm* and velocity *v* + *dv*. The products of mass –*dm* are ejected with a velocity $v - v_r$. (Note *dm* is negative). By conservation of momentum $mv = (m + dm)(v + dv) - dm(v - v_r)$, which simplifies to first order $0 = mdv + v_r dm$.

Hence $\int_{v_1}^{v_2} \frac{dv}{v_r} = -\int_{m_1}^{m_2} \frac{dm}{m}$, or $v_2 = v_1 + v_r \ln \frac{m_1}{m_2}$.

4 (a) KE = 2.09×10^{-20} J, and photon energy is 5.68×10^{-19} J

(b) Momentum is conserved, the initial momentum is $2m_{Cl}v_x\mathbf{i}$, one atom is scattered with momentum $m_{Cl}v_y\mathbf{j}$, thus $2m_{Cl}v_x\mathbf{i} = m_{Cl}v_y\mathbf{j} + \mathbf{p}$ so that $\mathbf{p} = 2m_{Cl}v_x\mathbf{i} - m_{Cl}v_y\mathbf{j}$, and $\mathbf{v} = 2v_x\mathbf{i} - v_y\mathbf{j} = (1200, -1600)$ m s^{-1}

(c) The atoms have KE 7.43×10^{-20} J and 1.16×10^{-19} J, the total is 1.90×10^{-19} J.

(d) Binding energy for the bond that has been broken.

5 Momentum is conserved, hence $2mv_f = mv$, so $v_f = v/2$

6 Momentum is conserved, and is equal to (7.37, 6.90) × 10^{-23} kg m s^{-1}. The velocity is obtained by dividing this vector by the mass of the complex, and is (249, 234) m s^{-1}. (The speed is 342 m s^{-1}.)

Elastic collisions.

7 (a) A collision in which both momentum and kinetic energy are conserved.

(b) (i) Denote the incident velocity of the neutron by **u**, before the collision the momentum of the neutron plus the C atom is $m_n\mathbf{u}$ because the C atom is stationary, and the velocity of the neutron relative to the C atom is **u**. After the collision the total momentum is unchanged, i.e. $m_n\mathbf{v}_n + m_C\mathbf{v}_C = m_n\mathbf{u}$ and the relative velocity will have changed sign, $\mathbf{v}_n - \mathbf{v}_C = -\mathbf{u}$. Solving,

$\mathbf{v}_n = -\frac{(m_C - m_n)}{(m_n + m_C)}\mathbf{u} = -2.2\times10^7\ \text{m s}^{-1}$ and $\mathbf{v}_C = \frac{2m_n}{(m_C + m_n)}\mathbf{u} = 4\times10^6\ \text{m s}^{-1}$, So the

(ii) The initial and final energies of the neutron are 5.6 × 10^{-13} J and 4.0 × 10^{-13} J, so the energy lost is 1.6 × 10^{-13} J (28% of the total).

(c) The maximum energy loss will be all the energy of the neutron. This will only be possible if $\frac{(m_X - m_n)}{(m_n + m_X)} = 0$, i.e. the nucleus should have a mass of 1 unit (which is why H atoms in water are widely used for this purpose).

8 (a) The balls have the same mass, momentum is conserved. Let the initial velocity of the cue ball be **u** in the *x*-direction, and its scattering speed and angle be v_c and θ, respectively. And let the scattering speed and angle of the object ball be v_o and ϕ, respectively.

Before the collision the total momentum vector is $\mathbf{p} = mu(1,0)$ and the relative velocity is (cue – object) $\mathbf{u}_{rel} = u(1,0)$.

After the collision the momentum is unchanged and the relative velocity has the same magnitude but a different direction. Hence we get the equations

$\mathbf{p} = mu(1,0) = mv_c(\cos\theta, \sin\theta) + mv_o(\cos\phi, \sin\phi)$ and

$$\left|\mathbf{v}_{rel}\right| = \left|v_c(\cos\theta,\sin\theta) - v_o(\cos\phi,\sin\phi)\right| = \left|(v_c\cos\theta - v_o\cos\phi, v_c\sin\theta - v_o\sin\phi)\right| = \left|\mathbf{u}_{rel}\right| = u$$

Conservation of momentum gives

$$v_c\cos\theta + v_o\cos\phi = u$$

$$v_c\sin\theta + v_o\sin\phi = 0$$

which can be used to substitute for the properties of the object ball giving,

$\left|(2v_c\cos\theta - u, 2v_c\sin\theta)\right| = u \Rightarrow v_c = u\cos\theta$. And now that v_c is known v_o can be found

$$\begin{aligned} v_o\cos\phi &= u - v_c\cos\theta \\ v_o\sin\phi &= -v_c\sin\theta \end{aligned}$$ so that $v_o^2 = u^2 - u^2\cos^2\theta$, or equivalently $v_o = u\sin\theta$,

and finally $\tan\phi = -\cot\theta$.

Putting in the numbers the cue ball is scattered with a speed of 4 m s^{-1} at an angle of 37°, and the object ball with a speed of 3 4 m s^{-1} at an angle of –53°. Note that the balls are moving in perpendicular directions. This is implied by the relationship between the two angles.

(b) Before the collision the total momentum vector is $\mathbf{p} = \frac{1}{2}mu(1,0)$ and the relative velocity is (cue – object) $\mathbf{u}_{rel} = u(1,0)$.

Hence we get the equations $\mathbf{p} = \frac{1}{2}mu(1,0) = \frac{1}{2}mv_c(\cos\theta,\sin\theta) + mv_o(\cos\phi,\sin\phi)$ and

$\left|\mathbf{v}_{rel}\right| = \left|v_c(\cos\theta,\sin\theta) - v_o(\cos\phi,\sin\phi)\right| = \left|(v_c\cos\theta - v_o\cos\phi, v_c\sin\theta - v_o\sin\phi)\right| = \left|\mathbf{u}_{rel}\right| = u$

Conservation of momentum gives

$$\begin{aligned} \tfrac{1}{2}v_c\cos\theta + v_o\cos\phi &= \tfrac{1}{2}u \\ \tfrac{1}{2}v_c\sin\theta + v_o\sin\phi &= 0 \end{aligned}$$

which can be used to substitute for the properties of the object ball giving,

$\left|(\frac{3}{2}v_c\cos\theta - \frac{1}{2}u, v_c\frac{3}{2}v_c\sin\theta)\right| = u \Rightarrow 3v_c^{\ 2} - 2uv_c\cos\theta - u^2 = 0$, a quadratic whose solution is

$v_c = \frac{1}{3}u\left(\cos\theta + \sqrt{3 + \cos^2\theta}\right) = 4.51 \text{ m s}^{-1}$.

And now that v_c is known v_o can be found

$$\begin{aligned} 2v_o\cos\phi &= u - v_c\cos\theta \\ 2v_o\sin\phi &= -v_c\sin\theta \end{aligned}$$ so that $v_o^2 = \frac{1}{4}\left(u^2 - 2uv_c\cos\theta + v_c^2\right) = 1.53 \text{ m s}^{-1}$,

and finally $\tan\phi = \dfrac{-v_c\sin\theta}{u - v_c\cos\theta} \Rightarrow \theta = -63°$.

9 At the closest approach the protons turn round, and so are momentarily stationary. All their KE has been converted to PE of Coulombic repulsion, i.e.

$$m_p v^2 = \frac{e^2}{4\pi\varepsilon_0 r} \Rightarrow r = \frac{e^2}{4\pi\varepsilon_0 m_p v^2} = 3.47\ \mu\text{m}$$

It is better to use the relative motion to analyse this problem, because it is easy to apply to the case where the speeds are not equal. $\dfrac{1}{2}\mu v_{rel}^2 = \dfrac{e^2}{4\pi\varepsilon_0 r}$. The result is the same.

10 (*a*) At closest approach the relative KE is zero, so the initial relative KE is all converted to PE.

$$\frac{1}{2}\mu u_{rel}^2 = 4\varepsilon\left(\left(\frac{\sigma}{r_0}\right)^{12} - \left(\frac{\sigma}{r_0}\right)^{6}\right) = 4\varepsilon\left(x^2 - x\right) \Rightarrow x^2 - x - \frac{\mu u_{rel}^2}{8\varepsilon} = 0 \Rightarrow x = \frac{1}{2}\left(1 + \sqrt{1 + \frac{\mu u_{rel}^2}{2\varepsilon}}\right) = 1.3$$

hence $r/\sigma = 0.957$ and r = 3.25 Å.

(b) At closest approach the relative velocity is zero so both atoms are travelling with the same speed, the centre of mass speed, 200 m s^{-1}.

(c) The repulsive force is the derivative of the PE,

$F = -\frac{dV}{dr} = \frac{4\varepsilon}{r}\left(12\left(\frac{\sigma}{r}\right)^{12} - 6\left(\frac{\sigma}{r}\right)^{6}\right) = 260\text{ pN}$, thus the relative acceleration is $F/\mu = 7.88\times 10^{15}\text{ m s}^{-2}$. The accelerations of each atom are equal and opposite and half of this.

(d) The momentum is conserved and the final relative velocity will be equal and opposite to the initial relative velocity. Thus $u_A + u_B = v_A + v_B$ and $u_A - u_B = -v_A + v_B$. Hence $v_A + v_B = 400\text{ m s}^{-1}$ and $-v_A + v_B = 400\text{ m s}^{-1}$, solving, $v_B = 400\text{ m s}^{-1}$ and $v_A = 0\text{ m s}^{-1}$. The incident particle transfers all its energy to the target.

Kinetic theory of pressure and effusion (application of mechanics).

11 (a) 297 m s^{-1}. (Common errors are to use g instead of kg for mass, and to forget that nitrogen is diatomic).

(b) In time *dt* all molecules in the cylinder of length *v dt* and cross sectional area *A* travelling towards the wall will hit it, the volume of the cylinder is $Av\,dt$, and it therefore contains $NAv\,dt/2$ particles (half molecules in the volume travelling in each direction). Thus in time *t* there will be $NAvt/2 = NAt\sqrt{RT/M}/2$ collisions.

(c) Each molecule changes momentum from *mv* to –*mv*. The total rate of change of momentum of gas molecules is therefore $-mNART/M$, and the force on the wall will be equal and opposite, $F = mNART/M$, so the pressure is $p = mNRT/M = NRT/N_A = nRT/V$

(d) By symmetry, gases are isotropic, so the mean square velocity component in any direction must be the same.

$$p = \frac{1}{3} Nm\overline{c^2}$$

12 (a) $Nf(v_x)dv_x$

(b) $Nf(v_x)dv_x \times Av_x dt$

(c) Rate of change of molecular momentum for this subgroup is $Nf(v_x)dv_x \times Av_x \times (-2mv_x) = -2NmAv_x^2 f(v_x)dv_x$, hence contribution to force is $2NmAv_x^2 f(v_x)dv_x$

(d) $F = 2NA\int_0^\infty v_x^2 f(v_x)dv_x$. But because the pdf is an even function of v_x and the integrals over negative and positive v_x are equal, hence $F = NmA\int_{-\infty}^{\infty} v_x^2 f(v_x)dv_x = NmA\overline{v_x^2}$.

A more sophisticated version is possible using the full 3d probability density for the velocity.